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On the supersolvability of a finite group by the sum of subgroup orders

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Let G be a finite group and $\sigma_1(G) = \frac{1}{|G|}\sum_{H\leq G}|H|$. In this paper, we prove that if $\sigma_1(G) < 2 + \frac{11}{|G|}$, then G is supersolvable. In particular, some new characterizations of the well-known groups $\mathbb{Z}_2 \times \mathbb{Z}_4$ and A_4 are obtained. We also show that $\sigma_1(G) < c$ does not imply the supersolvability of G for no constant $c \in (2, \infty)$.

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1. Introduction

Given a finite group G, we consider the function

$$\sigma_1(G) = \frac{1}{|G|}\sum_{H\leq G}|H|,$$

studied in our previous papers [7, 10, 11]. Recall some basic properties of σ_1:

- if G is cyclic of order n and $\sigma(n)$ denotes the sum of all divisors of n, then $\sigma_1(G) = \frac{\sigma(n)}{n}$;
- σ_1 is multiplicative, i.e. if G_i, $i = 1, 2, \ldots, m$, are finite groups of coprime orders, then $\sigma_1(\prod_{i=1}^m G_i) = \prod_{i=1}^m \sigma_1(G_i)$;
- $\sigma_1(G) \geq \sigma_1(G/H) + \frac{1}{(G:H)}(\sigma_1(H) - 1) \geq \sigma_1(G/H)$, for all $H \trianglelefteq G$.

The function σ_1 was used to provide some criteria for a group to belong to one of the well-known classes of groups. For instance, if $\sigma_1(G) \leq 2$ then G is cyclic of deficient or perfect order by [7], while if $\sigma_1(G) \leq \frac{117}{20}$ then G is solvable by [4, 11]. Also, we note that there is no constant $c \in (2, \infty)$ such that $\sigma_1(G) < c$ implies the nilpotency of G, but this can be obtained from the condition $\sigma_1(G) < 2 + \frac{4}{|G|}$ (see [10]).

In this paper, we will first determine the structure of finite groups G satisfying $\sigma_1(G) < 3$. This will allow us to prove that if $\sigma_1(G) < 2+\frac{11}{|G|}$ then G is supersolvable. Moreover, A_4 is the unique finite non-nilpotent group G such that $\sigma_1(G) = 2 + \frac{11}{|G|}$. Then we will show that $\sigma_1(G) < c$ does not imply the supersolvability of G for no constant $c \in (2, \infty)$.

For the proof of our results, we need the following two theorems (see [6, 8, Theorem 1]).

Theorem 1.1. *Suppose every non-normal maximal subgroup of a finite group G has the same order. Then G is solvable and the nilpotent length of G is at most two.*

Theorem 1.2. *A finite group G contains a cyclic maximal subgroup if and only if it is of one of the following types*:

(a) $G = P \times K_1$, *where K_1 is an arbitrary finite cyclic group and P is a Sylow p-subgroup of G containing a cyclic maximal subgroup.*
(b) $G = (K_2 \rtimes P) \times K_1$, *where K_1 is an arbitrary finite cyclic group which is a Hall subgroup of G, P is a Sylow p-subgroup of G containing a cyclic maximal subgroup, $K_2 \rtimes P$ is non-nilpotent, and the centralizer $C_P(K_2)$ is a maximal cyclic subgroup of P.*
(c) $G = (P \rtimes K_2) \times K_1$, *where K_1 is an arbitrary finite cyclic group which is a Hall subgroup of G and $G_1 = P \rtimes K_2$ is a non-nilpotent group satisfying the following conditions*: *P is a Sylow p-subgroup of G_1, $C_P(K_2) \supseteq \Phi(P)$, and $C_P(K_2)$ is a cyclic invariant subgroup of G_1 such that $K_2C_P(K_2)/C_P(K_2)$ is a maximal subgroup of $G_1/C_P(K_2)$.*

We also need two auxiliary results, taken from [7, 11], respectively.

Lemma 1.3. *Let G be a finite group and $[M]$ be a conjugacy class of non-normal maximal subgroups of G. Then*

$$\sum_{H\in[M]} |H| = |G|.$$

Lemma 1.4. *Let G be a finite group. Then*

$$\sum_{H\leq G,\, H=\text{cyclic}} |H| \geq |G|.$$

Note that similar problems for some other functions related to the structure of a finite group G, for example for the function $\psi(G) = \sum_{x\in G} o(x)$ (where $o(x)$ denotes the order of the element x), have been recently investigated by many authors (see, e.g. [1–3]).

Most of our notations are standard and will not be repeated here. Basic definitions and results on groups can be found in [5]. For subgroup lattice concepts we refer the reader to [9].

2. Main Results

We start with the following important lemma.

Lemma 2.1. *Let G be a finite group with $\sigma_1(G) < 3$. Then either G is nilpotent or a group of type c) in Theorem* 1.2.

Proof. Assume that G is not nilpotent. If G has at least two conjugacy classes of non-normal maximal subgroups $[M_1]$ and $[M_2]$, then Lemma 1.3 leads to

$$\sigma_1(G) > \frac{1}{|G|}\left(|G| + \sum_{i=1}^{2}\sum_{H\in[M_i]}|H|\right) = \frac{1}{|G|}\left(|G| + 2|G|\right) = 3,$$

a contradiction. Thus G possesses a unique conjugacy class of non-normal maximal subgroups, say $[M]$. Moreover, M is cyclic. Indeed, if this is not true, then Lemmas 1.3 and 1.4 imply

$$\sigma_1(G) \geq \frac{1}{|G|}\left(|G| + \sum_{H\in[M]}|H| + \sum_{H\leq G,\, H=\text{cyclic}}|H|\right) \geq \frac{1}{|G|}\left(|G| + |G| + |G|\right) = 3,$$

contradicting again our hypothesis.

From Theorem 1.1, it follows that G is solvable of nilpotent length two. More precisely, by taking a prime p dividing $[G : M]$, we deduce that a Sylow p-subgroup P of G must be normal and a p-complement of G must be cyclic. This shows that G is a group of type c) in Theorem 1.2, completing the proof. $\square$

We are now able to prove our first main result.

Theorem 2.2. *Let G be a finite group. If $\sigma_1(G) < 2 + \frac{11}{|G|}$, then G is supersolvable.*

Proof. Since all groups of order ≤ 11 are supersolvable, we may assume that $|G| \geq 12$. Then $\sigma_1(G) < 3$ and therefore, by the above lemma, either G is nilpotent or $G = (P \rtimes K_2) \times K_1$ with P, K_1 and K_2 as in Theorem 1.2(c). In the first case we are done, while in the second case we may assume that $G = G_1 = P \rtimes K_2$ because

$$\sigma_1(G_1) = \sigma_1(G/K_1) \leq \sigma_1(G) < 2 + \frac{11}{|G|} \leq 2 + \frac{11}{|G_1|}. \tag{1}$$

Denote by $[M]$ the unique conjugacy class of non-normal maximal subgroups of G. Let $|P| = p^r$ and suppose that P is not cyclic. Then $r \geq 2$ and P contains at least

$p+1$ subgroups of order p^{r-1}. It follows that

$$\sigma_1(G) \geq \frac{1}{|G|}\left(|G| + \sum_{H\in[M]} |H| + \sum_{H\leq P} |H|\right) \geq 2 + \frac{1+p+\cdots+(p+1)p^{r-1}+p^r}{|G|}$$
$$\geq 2 + \frac{1+(p+1)p+p^2}{|G|} = 2 + \frac{1+p+2p^2}{|G|} \geq 2 + \frac{11}{|G|},$$

contradicting (1). Thus P is cyclic and consequently all Sylow subgroups of G are cyclic, implying that G is supersolvable. This completes the proof. □

Next, we will determine finite groups G such that $\sigma_1(G) = 2 + \frac{11}{|G|}$. First of all, we will focus on p-groups. We observe that for all primes p and all positive integers n we have

$$\sigma_1(\mathbb{Z}_{p^n}) = \frac{\sigma(p^n)}{p^n} = \frac{1+p+\cdots+p^n}{p^n} = 1 + \frac{1}{p-1} - \frac{1}{p^n(p-1)} < 2.$$

The following lemma deals with the case of non-cyclic p-groups.

Lemma 2.3. *Let G be a non-cyclic p-group of order p^n. Then*

(a) $\sigma_1(G) < 2 + \frac{11}{|G|}$ *if and only if either $n = 3$, $p = 2$ and $G = Q_8$, or $n = 2$, $p \in \{2, 3, 5, 7\}$ and $G = \mathbb{Z}_p \times \mathbb{Z}_p$;*
(b) $\sigma_1(G) = 2 + \frac{11}{|G|}$ *if and only if $n = 3$, $p = 2$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_4$.*

Proof. Since G contains at least $p+1$ subgroups of order p^{n-1}, one obtains

$$2 + \frac{11}{p^n} \geq \sigma_1(G) \geq \frac{1+p+\cdots+(p+1)p^{n-1}+p^n}{p^n} = 2 + \frac{1+p+\cdots+p^{n-1}}{p^n},$$

that is $1+p+\cdots+p^{n-1} \leq 11$. Thus either $n = 3$ and $p = 2$, or $n = 2$ and $p \in \{2, 3, 5, 7\}$. The conclusion follows by checking the corresponding groups. □

Our second main result is stated as follows.

Theorem 2.4. *Let G be a finite non-cyclic group. If $\sigma_1(G) = 2 + \frac{11}{|G|}$, then either $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ or $G = A_4$.*

Proof. We can easily see that $G = \mathbb{Z}_2 \times \mathbb{Z}_4$ is the unique group G of order ≤ 11 satisfying $\sigma_1(G) = 2 + \frac{11}{|G|}$. Assume that $|G| \geq 12$. Then $\sigma_1(G) < 3$ and therefore, by Lemma 2.1, either G is nilpotent or $G = (P \rtimes K_2) \times K_1$ with P, K_1 and K_2 as in Theorem 1.2(c).

In the first case, let $G = G_1 \times \cdots \times G_k$ be the decomposition of G as a direct product of Sylow p-subgroups. For $k = 1$, we have no solution G of $\sigma_1(G) = 2 + \frac{11}{|G|}$ by Lemma 2.3(b). For $k \geq 2$, we get $\sigma_1(G_i) < 2 + \frac{11}{|G_i|}$, $\forall\, i = 1, \ldots, k$. Then

Lemma 2.3(a), shows that every G_i is of one of the following types: Q_8, $\mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}$ or $\mathbb{Z}_{p_i^{n_i}}$ for some prime p_i and some positive integer n_i. Since

$$\sigma_1(Q_8) = \frac{23}{8} > 2,\ \sigma_1(\mathbb{Z}_{p_i} \times \mathbb{Z}_{p_i}) = 2 + \frac{p_i+1}{p_i^2} > 2, \quad \forall\, i = 1, \ldots, k$$

and

$$\sigma_1(G) = \sigma_1(G_1) \cdots \sigma_1(G_k),$$

we deduce that at most one of G_i's is non-cyclic. Suppose that $G_1 = Q_8$ and $G_i = \mathbb{Z}_{p_i^{n_i}}$, for all $i = 2, \ldots, k$. Then our condition becomes

$$\frac{23}{8}\,\frac{\sigma(p_2^{n_2})}{p_2^{n_2}} \cdots \frac{\sigma(p_k^{n_k})}{p_k^{n_k}} = 2 + \frac{11}{8p_2^{n_2} \cdots p_k^{n_k}},$$

for example,

$$23\,\sigma(p_2^{n_2}) \cdots \sigma(p_k^{n_k}) = 16p_2^{n_2} \cdots p_k^{n_k} + 11,$$

a contradiction since the left side is greater than the right side. Similarly, if we suppose $G_1 = \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$ and $G_i = \mathbb{Z}_{p_i^{n_i}}$, for all $i = 2, \ldots, k$, then we obtain

$$\frac{2p_1^2 + p_1 + 1}{p_1^2}\,\frac{\sigma(p_2^{n_2})}{p_2^{n_2}} \cdots \frac{\sigma(p_k^{n_k})}{p_k^{n_k}} = 2 + \frac{11}{p_1^2 p_2^{n_2} \ldots p_k^{n_k}},$$

for example,

$$(2p_1^2 + p_1 + 1)\,\sigma(p_2^{n_2}) \cdots \sigma(p_k^{n_k}) = 2p_1^2 p_2^{n_2} \ldots p_k^{n_k} + 11,$$

a contradiction since the left side is greater than the right side. Thus every G_i is cyclic, and so is G, contradicting our hypothesis.

In the second case, a similar reasoning shows that we have no solution $G = (P \rtimes K_2) \times K_1$ of $\sigma_1(G) = 2 + \frac{11}{|G|}$ with $K_1 \neq 1$. Consequently, we may assume $G = P \rtimes K_2$. Let $|P| = p^r$ and $|K_2| = n$. Note that $p \nmid n$ and $n \mid |\mathrm{Aut}G|$. As in the proof of Theorem 2.2, we get

$$2 + \frac{11}{|G|} = \sigma_1(G) \geq 2 + \frac{\sum\limits_{H \leq P} |H|}{|G|}$$

and thus

$$\sum_{H \leq P} |H| \leq 11.$$

It follows that either $r = 2$, $p = 2$ and $P \in \{\mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2\}$, or $r = 1$, $p \in \{2, 3, 5, 7\}$ and $P = \mathbb{Z}_p$.

If $P = \mathbb{Z}_4$, then $n \mid 2$, a contradiction. If $P = \mathbb{Z}_2 \times \mathbb{Z}_2$, then $n \mid 6$ and so $n = 3$; thus $G = A_4$, which is a solution of $\sigma_1(G) = 2 + \frac{11}{|G|}$. If $P = \mathbb{Z}_p$ with $p \in \{2, 3, 5, 7\}$, then n divides 1, 2, 4 or 6, respectively. We deduce that the unique possibilities are

$$(p, n) \in \{(3, 2), (5, 2), (5, 4), (7, 2), (7, 3), (7, 6)\}.$$

By checking all these semidirect products $G = \mathbb{Z}_p \rtimes \mathbb{Z}_n$ we find no other solution of $\sigma_1(G) = 2 + \frac{11}{|G|}$. Hence in this case the unique solution is $G = A_4$, as desired. $\square$

Obviously, Theorem 2.4 leads to the following characterizations of $\mathbb{Z}_2 \times \mathbb{Z}_4$ and A_4.

Corollary 2.5. $\mathbb{Z}_2 \times \mathbb{Z}_4$ *is the unique finite non-cyclic nilpotent group* G *satisfying* $\sigma_1(G) = 2 + \frac{11}{|G|}$, *while* A_4 *is the unique finite non-nilpotent group* G *satisfying* $\sigma_1(G) = 2 + \frac{11}{|G|}$.

Finally, we remark that $\sigma_1(G) < \frac{31}{12} = \sigma_1(A_4)$ does not imply the supersolvability of G, a counterexample being SmallGroup$(56, 11)$ (also known as the Frobenius group F_8). In fact, we are able to prove that there is no constant $c \in (2, \infty)$ such that $\sigma_1(G) < c$ implies the supersolvability of G.

Theorem 2.6. *There are sequences of finite non-supersolvable groups* $(G_n)_{n\in\mathbb{N}}$ *such that* $\sigma_1(G_n)$ *approaches* 2 *monotonically from above as* n *tends to infinity.*

Proof. Let $(q_n)_{n\in\mathbb{N}}$ be the sequence of odd prime numbers. By Dirichlet's theorem we deduce that for every $n \in \mathbb{N}$ there is a prime p_n such that $q_n \mid p_n - 1$. Let G_n be the unique non-abelian group of order $p_n^2 q_n$. Clearly, G_n is not supersolvable. It contains one subgroup of order 1, $p_n + 1$ subgroups of order p_n, one subgroup of order p_n^2, p_n^2 subgroups of order q_n, and one subgroup of order $p_n^2 q_n$. Then

$$\sigma_1(G_n) = \frac{1 + (p_n + 1)p_n + p_n^2 + p_n^2 q_n + p_n^2 q_n}{p_n^2 q_n} = 2 + \frac{2 + \frac{1}{p_n} + \frac{1}{p_n^2}}{q_n},$$

which approaches 2 monotonically from above as n tends to infinity, as desired. $\square$

We end this paper by formulating a natural problem concerning the above study.

Open problem. Find all finite cyclic groups G satisfying $\sigma_1(G) = 2 + \frac{11}{|G|}$.

Note that this is equivalent to finding all positive integers $n = p_1^{n_1} \dots p_k^{n_k}$ such that

$$(1 + p_1 + \cdots + p_1^{n_1}) \dots (1 + p_k + \cdots + p_k^{n_k}) = 2p_1^{n_1} \dots p_k^{n_k} + 11.$$

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